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Exact Results in Noncommutative $\mathcal{N}=2$ Supersymmetric Gauge Theories

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Abstract: We study the low-energy dynamics of noncommutative $\mathcal{N}=2$ supersymmetric $U(N)$ Yang-Mills theories in the Coulomb phase. Exact results are derived for the leading terms in the derivative expansion of the Wilsonian effective action. We find that in the infrared regime the $U(1)$ subgroup decouples, and the remaining $SU(N)$ is described by the ordinary commutative Seiberg-Witten solution. IR/UV mixing is present in the $U(1)$, but not in $SU(N)$. Our analysis is based on explicit perturbative and multi-instanton calculations.

Keywords: Noncommutative Gauge Theories, Supersymmetry, Instantons.

1. Introduction

Recently there has been a lot of interest in gauge theories on noncommutative spaces. One of the reasons is the natural appearance of noncommutativity in the framework of string theory and D-branes [1–3]. Noncommutative gauge theories are also fascinating on their own right mostly due to a new interplay between the infrared (IR) and the ultraviolet (UV) degrees of freedom discovered in [4]. It is also known that this IR/UV mixing does not occur in $\mathcal{N}=4$ supersymmetric noncommutative gauge theories [5]. This is supported by the $\mathcal{N}=4$ gauge/supergravity correspondence discussed in [6, 7].

In this paper we analyse the leading terms in the derivative expansion of the Wilsonian effective action for $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theories[1] on noncommutative space with $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. Specifically we will concentrate on the terms with at most two space-time derivatives and/or not more than four fermions. Such terms in the Wilsonian Lagrangian will be denoted $\mathcal{L}_{\rm eff}$. We will demonstrate that for Wilsonian momentum scales k^2 below the noncommutativity mass-scale, $k^2 \ll M_{NC}^2 \sim \theta^{-1}$, the $U(1)$ degrees of freedom decouple from the $SU(N)$ fields and

$$\mathcal{L}_{\rm eff}^{U(N)}(k) \ = \ \mathcal{L}_{\rm eff}^{U(1)}(k) \ + \ \mathcal{L}_{\rm eff}^{SU(N)}(k) \ . \tag{1.1}$$

We will concentrate on the Coulomb branch of the theory and parametrize the vacuum expectation values (VEVs) of the adjoint scalar field via

$$\langle\varphi\rangle \ = \ {\rm diag}({\rm v}_1, \dots, {\rm v}_N) \ . \tag{1.2}$$

Without loss of generality this matrix of VEVs can be chosen traceless, $\sum_{u=1}^N {\rm v}_u = 0$, since the $U(1)$-part of the scalar VEV, $\langle A\rangle = {\rm V}_{\rm tr}\,{\rm diag}(1, \dots, 1)$, breaks no symmetries and does not play any role in the dynamics of the theory. In fact, as noticed in [8], the transformation ${\rm V}_{\rm tr} \to {\rm V}_{\rm tr} + const$ is a symmetry of the theory and ${\rm V}_{\rm tr}$ is not a coordinate on the quantum moduli space.

Since the noncommutative $U(1)$ decouples from $SU(N)$, it can be analysed separately on its own right. Such an analysis of the Wilsonian action $\mathcal{L}_{\rm eff}^{U(1)}(k)$ was carried out in the earlier work [9] at the one-loop level, where it was found that, due to the IR/UV mixing, the $U(1)$ $\mathcal{N}=2$ theory remains noncommutative even in the IR region, $k \ll M_{NC}$, and arbitrarily weakly coupled as $k^2 \to 0$, justifying the one-loop analysis. The approach of [9] determines the RG flow of the Wilsonian $U(1)$ coupling constant, $g'^2_{\rm eff}(k)$, in such a way that

$$\frac{1}{g'^2_{\rm eff}(k)} \to \frac{1}{8\pi^2}\log k^2 \ , \qquad {\rm as}\ k^2 \to \infty \ , \tag{1.3}$$

$$\frac{1}{g'^2_{\rm eff}(k)} \to \frac{1}{8\pi^2}\log \frac{1}{k^2} \ , \qquad {\rm as}\ k^2 \to 0 \ . \tag{1.4}$$

[1] This week similar issues were addressed from a different perspective in the interesting work [8].

Thus, the noncommutative $U(1)$ theory is asymptotically free and weakly coupled in the UV region, and it changes its behaviour to a screening regime at $k \sim M_{NC}$, and becomes arbitrarily weakly coupled in the IR. This running of g'^2_{eff} is strikingly different from the coupling of the ordinary $\mathcal{N}=2$ commutative $U(1)$ which is k-independent, *i.e.* does not run. The corresponding 2-derivative Wilsonian action reads:

$$\mathcal{L}^{U(1)}_{\text{eff}}(k) \;=\; -\frac{1}{2g'^2_{\text{eff}}(k)}\;\text{Tr}\left(F^{U(1)}_{\mu\nu} \star F^{U(1)}_{\mu\nu}\right) \;+\; \ldots \;, \tag{1.5}$$

where the dots stand for the $\mathcal{N}=2$ superpartners of the $U(1)$ gauge kinetic term, and the star-product is defined in the standard way as

$$(\phi \star \chi)(x) \equiv \phi(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \chi(x)\;. \tag{1.6}$$

Let us now discuss the $SU(N)$ degrees of freedom. The Higgs VEVs (1.2) spontaneously break the gauge symmetry $SU(N) \to U(1)^{N-1}$ by giving masses to the W-bosons and their superpartners, $M_W \propto |\mathrm{v}_u - \mathrm{v}_v|$, and leave the $N-1$ fields in the Cartan subalgebra of $SU(N)$ massless. At momentum scales $k < M_W$ the massive degrees of freedom will be integrated out leading to the Wilsonian action, $\mathcal{L}^{[N-1]}_{\text{eff}}(k)$, of the $N-1$ massless photons and their superpartners. One of the principal results of this paper is the fact that $\mathcal{L}^{[N-1]}_{\text{eff}}$ actually does not depend on k for $k < M_W$. It will turn out that the running $SU(N)$ coupling constant will freeze at the momentum scale $k = M_W$ and the resulting $(N-1)\times(N-1)$ matrix of coupling constants in the $U(1)^{N-1}$ low-energy theory will not depend on k. Instead it will depend on the VEVs $\mathrm{v}_1, \ldots \mathrm{v}_N$ which set the values of the W-masses, where the freezing occurs. In other words, $\mathrm{v}_1, \ldots \mathrm{v}_N$ will parametrize the vacuum moduli space of the $\mathcal{N}=2$ noncommutative $SU(N)$ theory similarly to the situation in the ordinary commutative $\mathcal{N}=2$ scenario of Seiberg and Witten [10].

This relation between the noncommutative and the commutative $\mathcal{N}=2$ $SU(N)$ theories in the Coulomb branch is more than just an analogy, it is an equivalence.[2] We will demonstrate below that no IR/UV mixing occurs in the $SU(N)$ sector in perturbation theory and non-perturbatively. This is unexpected for noncommutative theories with $\mathcal{N}<4$ supersymmetries. The fact is that the IR/UV mixing in $U(N)$ occurs only in the $U(1)$ part of the theory which decouples from the $SU(N)$ degrees of freedom and can be neglected in the IR. This $U(1)$ theory is massless and can be characterized as being in the 'noncommutative Coulomb phase'. The low-energy $U(1)^{N-1}$ theory $\mathcal{L}^{[N-1]}_{\text{eff}}$ can be formulated in superspace similarly to the ordinary theory [11, 12]. Hence, it can be written in terms of the holomorphic prepotential $\mathcal{F}(\Phi)$

$$\mathcal{L}^{[N-1]}_{\text{eff}} \;=\; \text{Im}\,\frac{1}{4\pi}\left[\int d^4\theta\, \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi_I} \star \bar{\Phi}_I \;+\; \frac{1}{2}\int d^2\theta\, \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_I\, \partial \Phi_J} \star W_I \star W_J\right]. \tag{1.7}$$

[2]Our analysis will be valid of course only for the leading term in the derivative expansion of the low-energy effective action.

Here Φ_I and W_I are the $\mathcal{N}=1$ chiral $U(1)^{N-1}$ superfields containing the Ith massless Higgs boson and the Ith photon field strength, respectively, and $I, J = 1, \ldots, N-1$. The prepotential $\mathcal{F}$ is a holomorphic function of the $\mathcal{N}=1$ chiral superfields Φ_I and in general it can also depend on the noncommutativity parameters $\theta^{\mu\nu}$. For the purposes of the low-energy theory we will set the superfields Φ_I in the prepotential equal to their VEVs v_I. The matrix of the Wilsonian coupling constants of the $U(1)^{N-1}$ theory is determined via

$$\frac{\partial^2 \mathcal{F}(\mathrm{v})}{\partial \mathrm{v}_I \, \partial \mathrm{v}_J} \;=\; \tau(\mathrm{v})_{IJ} \;=\; \frac{4\pi i}{g^2_{\text{eff}}(\mathrm{v})_{IJ}} \;+\; \frac{\vartheta_{\text{eff}}(\mathrm{v})_{IJ}}{2\pi} \;, \tag{1.8}$$

where ϑ_{eff} is the effective theta-angle.

It is well-known [13] that the prepotential is completely specified by a perturbative one-loop contribution, and an infinite multi-instanton expansion

$$\mathcal{F}(\mathrm{v},\theta) \;=\; \mathcal{F}_{\text{1-loop}}(\mathrm{v},\theta) \;+\; i\sum_{k=1}^{\infty} \mathcal{F}_k(\mathrm{v},\theta) \;, \tag{1.9}$$

where $\mathcal{F}_k$ denotes the contribution of the k-instanton sector. In the rest of this paper we will calculate the perturbative contribution $\mathcal{F}_{\text{1-loop}}$, and deduce all the multi-instanton contributions to the prepotential from the field theory side. We will find that $\mathcal{F}$ does not depend on θ and agrees precisely with the ordinary commutative Seiberg-Witten prepotential.

The rest of the paper is organized as follows. In Section 2 we study the $N^2 \times N^2$ matrix of Wilsonian coupling constants of the noncommutative $U(N)$ theory at one-loop level using the background field method. We find that after decoupling of massive degrees of freedom this matrix factorizes in the IR region,

$$\frac{1}{g^2(k)}_{[N^2]\times[N^2]} \;\to\; \frac{1}{g^2_{\text{eff}}(k)}_{[N]\times[N]} \;=\; \frac{1}{g'^2_{\text{eff}}(k)} \;\oplus\; \frac{1}{g^2_{\text{eff}}(\mathrm{v})}_{[N-1]\times[N-1]} \tag{1.10}$$

which corresponds to the perturbative[3] decoupling of $U(N) \to U(1) \times SU(N)$. The IR/UV mixing effects will be present in the $U(1)$ coupling and absent in the $SU(N)$ coupling constant. The latter will be frozen at the W-mass scale v, and its dependence on v will be exactly the same as in the commutative $SU(N)$ theory, hence it will match precisely with the commutative Seiberg-Witten prepotential $\mathcal{F}_{\text{1-loop}}$.

In Section 3 we consider instanton and anti-instanton contributions to the low-energy effective action in the noncommutative $U(N)$ gauge theory. We give a general argument that all multi-instanton contributions to the prepotential in the noncommutative case do not depend on the noncommutativity parameter and agree with the ordinary commutative contributions.

[3] It will follow from the analysis in Section 3 that this decoupling is also respected by instantons.

In Section 4 this general multi-instanton argument is explicitly verified with a detailed one-instanton calculation.

Note on Conventions

We introduce anti-hermitian generators of $U(N)$ as t^A, $A = (0, a)$, where $a = 1, \ldots, N^2 - 1$ labels the $SU(N)$ generators, and $t^0 = (1/i\sqrt{2N})\mathbb{1}_N$. Then

$$\mathrm{Tr}(t^A t^B) = -\frac{\delta^{AB}}{2} \quad . \tag{1.11}$$

The generators satisfy

$$[t^A, t^B] = f^{ABC} t^C \ , \tag{1.12}$$

$$\{t^A, t^B\} = -\frac{\delta^{AB}}{N} - i d^{ABC} t^C \quad . \tag{1.13}$$

f^{ABC} (d^{ABC}) is completely antisymmetric (symmetric) in its indices; f^{abc}, d^{abc} are the same as in $SU(N)$, and $f^{0bc} = 0$, $d^{0BC} = \sqrt{\frac{2}{N}}\delta_{BC}$, $\delta^{00a} = 0$, $d^{000} = \sqrt{\frac{2}{N}}$.

Given an arbitrary four-vector, we will also use the notation $\tilde{k}_\mu \equiv \theta_{\mu\nu} k_\nu$.

The Euclidean σ_μ and $\bar{\sigma}_\mu$ matrices are defined as $\sigma_\mu = (\mathbb{1}_{2\times 2}, i\sigma^m)$ and $\bar{\sigma}_\mu = (\mathbb{1}_{2\times 2}, -i\sigma^m)$ where σ^m are the three Pauli matrices. We will also use $\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) = i\eta^a_{\mu\nu}\sigma^a$, and $\bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) = i\bar{\eta}^a_{\mu\nu}\sigma^a$, where $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are the 't Hooft symbols [14].

2. One-loop calculation of the effective action

In this Section we will apply the background field perturbation theory to noncommutative $U(N)$. Our discussion here follows closely the formalism introduced in [9], to which we refer the reader for further details. The gauge field A_μ is decomposed into a background field B_μ and a fluctuating quantum field N_μ,

$$A_\mu = B_\mu + N_\mu \quad , \tag{2.1}$$

where N_μ is a highly virtual field with momenta above the Wilsonian scale. The background field is slowly varying, but fully noncommutative. The effective action $S_{\text{eff}}(B)$ is obtained by functionally integrating over the fluctuating fields. Noncommutative gauge-invariance constrains the interactions which can be generated in this procedure. Therefore, the effective action will always contain the kinetic term

$$S_{\text{eff}}[B] \ni -\frac{1}{2g^2_{\text{eff}}} \int d^4x \ \mathrm{Tr}\left(F^{(B)}_{\mu\nu} \star F^{(B)}_{\mu\nu}\right) \quad . \tag{2.2}$$

The multiplicative coefficient on the right hand side is identified with the Wilsonian coupling constant at the corresponding momentum scale. In order to determine $g_{\rm eff}$ it is sufficient to consider the kinetic term $(\partial_\mu B_\nu - \partial_\nu B_\mu)^2$. In the effective Lagrangian, this term becomes

$$2\int \frac{d^4k}{(2\pi)^4} B^A_\mu(k) B^B_\nu(-k)\ \Pi^{AB}_{\mu\nu} \quad . \tag{2.3}$$

Equation (2.3) defines the *Wilsonian polarization tensor* $\Pi^{AB}_{\mu\nu}(k)$, which in the effective theory replaces the tree level transverse tensor $(k^2\delta_{\mu\nu} - k_\mu k_\nu)$. On general grounds, $\Pi^{AB}_{\mu\nu}(k)$ has the structure

$$\Pi^{AB}_{\mu\nu}(k) = \Pi^{AB}_1(k^2,\tilde{k}^2)(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \Pi^{AB}_2(k^2,\tilde{k}^2)\frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^4} \quad . \tag{2.4}$$

Here $\Pi^{AB}_1(k^2,\tilde{k}^2)$ determines the matrix of the Wilsonian couplings,

$$\left[\frac{1}{g^2_{\rm eff}(k)}\right]^{AB} = \frac{\delta^{AB}}{g^2_{\rm micro}} + 4\Pi^{AB}_1(k^2,\tilde{k}^2) \ . \tag{2.5}$$

The term in (2.4) proportional to $\tilde{k}_\mu \tilde{k}_\nu/\tilde{k}^4$ would not appear in ordinary commutative theories. It is transverse and has derivative dimension -2; therefore it is of leading order compared to the standard gauge-kinetic term (which has derivative dimension $+2$), and, most importantly, leads to an infrared singular behaviour. In [9] it was shown that Π_2 vanishes for all supersymmetric noncommutative $U(1)$ gauge theories (unbroken and softly broken), as was first discussed in [5]. Π_2 is an intrinsically noncommutative object and arises only from nonplanar diagrams, whereas Π_1 receives contribution from planar as well as from nonplanar diagrams.

The action functional which describes the dynamics of a spin-j noncommutative field in the representation $\mathbf{r}$ of the gauge group in the background of B_μ has the general form [9, 15]

$$\begin{aligned} S[\phi] &= -\int d^4x\ \phi_{m,a} \star \left(-D^2(B)\delta_{mn}\delta^{ab} + 2i(F^B_{\mu\nu})^{ab}\tfrac{1}{2}J^{\mu\nu}_{mn}\right) \star \phi_{n,b} \\ &\equiv -\int d^4x\ \phi_{m,a} \star [\Delta_{j,\mathbf{r}}]^{ab}_{mn} \star \phi_{n,b} \quad . \end{aligned} \tag{2.6}$$

Here a, b are indices of the representation $\mathbf{r}$ of noncommutative $U(N)$, $F^{ab} \equiv \sum_{A=1}^{N^2} F^A t^A_{ab}$, and m, n are spin indices and $J^{\mu\nu}_{mn}$ are the generators of the euclidean Lorentz group appropriate for the spin of ϕ:

$$\begin{aligned} J &= 0 && \text{for spin 0 fields,} \\ J^{\mu\nu}_{\rho\sigma} &= i(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma) && \text{for 4-vectors,} \\ [J^{\mu\nu}]_\alpha^{\ \beta} &= i\tfrac{1}{2}[\sigma^{\mu\nu}]_\alpha^{\ \beta} && \text{for Weyl fermions} \quad . \end{aligned} \tag{2.7}$$

At the one-loop level, the effective action is given by [9]

$$S_{\rm eff}[B] = -\frac{1}{2g^2}\int d^4x \;{\rm Tr} F^B_{\mu\nu}\star F^B_{\mu\nu} - \sum_{j,\mathbf{r}} \alpha_j \log\det{}_\star \Delta_{j,\mathbf{r}} \quad , \tag{2.8}$$

where the sum is extended to all fields in the theory, including ghosts and gauge fields. α_j is equal to $+1$ (-1) for ghost (scalar) fields and to $+1/2$ $(-1/2)$ for Weyl fermions (gauge fields). Functional star-determinants are computed by

$$\begin{aligned}\log\det{}_\star\Delta_{j,\mathbf{r}} \equiv& \log\det{}_\star(-\partial^2 + \mathcal{K}(B)_{j,\mathbf{r}})\\ &= \log\det{}_\star(-\partial^2) + {\rm tr}_\star \log(1+(-\partial^2)^{-1}\mathcal{K}(B)_{j,\mathbf{r}}) \;.\end{aligned} \tag{2.9}$$

The first term on the second line of (2.9) contributes only to the vacuum loops and will be dropped in the following. The second term on the last line of (2.9) has an expansion in terms of Feynman diagrams.

2.1 Feynman rules

Since our main target is the computation of the effective action for $\mathcal{N}=2$ noncommutative Super Yang-Mills, we restrict our attention to fields transforming according to the adjoint representation of the group $U(N)$. We start off our analysis considering the case of vanishing vacuum expectation value for scalar fields, postponing the discussion of spontaneously broken theories to Section 2.3.

As in [9], we rewrite $\Delta_{j,\mathbf{r}}$ acting on adjoint fields as

$$\begin{aligned}\Delta_{j,\mathbf{G}}\star\phi \equiv& -\partial^2\phi + \mathcal{K}(B)_{j,\mathbf{G}}\star\phi\\ &= -\partial^2\phi - [(\partial_\mu B_\mu),\phi]_\star - 2\,[B_\mu\partial_\mu,\phi]_\star - \big[B_\mu,[B_\mu,\phi]_\star\big]_\star + 2i\left(\tfrac{1}{2}J^{\mu\nu}\left[F^B_{\mu\nu},\phi\right]_\star\right) \quad .\end{aligned} \tag{2.10}$$

The main difference with respect to the $U(1)$ case considered in [9] is that now

$$[\phi_1,\phi_2]_\star = \left(-\frac{i}{2}[\phi_1^A,\phi_2^B]_\star d^{ABC} + \frac{1}{2}\{\phi_1^A,\phi_2^B\}_\star f^{ABC}\right)t^C \quad . \tag{2.11}$$

The Taylor expansion of the logarithm in (2.9) will involve the Feynman diagrams made from the three interaction vertices. The first one, the ϕ-B-ϕ vertex, follows from the second and the third term on the second line in (2.10),

$$-2\mathrm{Tr}\int d^4x\ \bar\phi\star[(\partial_\mu B_\mu)+2B_\mu\partial_\mu,\phi]_\star\ =\ \int\frac{d^4p'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\ (2\pi)^4\delta^{(4)}(p+q-p')$$
$$\bar\phi^A(p')B_\mu^B(q)\phi^C(p)\ \left[i(2p+q)_\mu(-d^{ABC}\sin\frac{q\tilde p}{2}+f^{ABC}\cos\frac{q\tilde p}{2})\right]\quad . \tag{2.12}$$

The second vertex ϕ-B-B-ϕ follows from the fourth term on the second line in (2.10),

$$-2\mathrm{Tr}\int d^4x\ \bar\phi\star\left[B_\mu,[B_\mu,\phi]_\star\right]_\star\ =$$
$$\int\frac{d^4p'}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\ (2\pi)^4\delta^{(4)}(p+q_1+q_2-p')\ \bar\phi^A(p')B_\mu^B(q_1)B_\nu^C(q_2)\phi^D(p)\delta_{\mu\nu}$$
$$(-d^{BHA}\sin\frac{p'\tilde q_1}{2}+f^{BHA}\cos\frac{p'\tilde q_1}{2})\,(d^{CDH}\sin\frac{q_2\tilde{p'}}{2}+f^{CDH}\cos\frac{q_2\tilde p}{2})\quad , \tag{2.13}$$

and finally the third vertex follows from the last term on the second line in (2.10):

$$2i\mathrm{Tr}\int d^4x\ \bar\phi J^{\mu\nu}\star[\partial_\mu B_\nu-\partial_\nu B_\mu,\phi]_\star\ =\ \int\frac{d^4p'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\ (2\pi)^4\delta^{(4)}(p+q-p')$$
$$\bar\phi^A(p')J^{\mu\nu}B_B^\nu(q)\phi^C(p)\ \left[2q_\mu(-d^{ABC}\sin\frac{q\tilde p}{2}+f^{ABC}\cos\frac{q\tilde p}{2})\right]\quad . \tag{2.14}$$

The first two vertices (2.12) and (2.13) are the standard Feynman vertices for noncommutative electrodynamics with an adjoint scalar field, and the third expression (2.14) is the so-called J-vertex, which arises in the background field method [15].

2.2 Planar and nonplanar contributions to the effective action

Expanding the logarithm in (2.9) to the second order in the background fields B_μ gives the Feynman graphs shown in Figures 1, 2 and 3, in which the J-vertices are depicted by a cross. Dimensional regularization is understood in all the integrals, and UV-divergences are removed with the supersymmetry-preserving $\overline{\mathrm{DR}}$-scheme [16].

The first Feynman graph (shown in Figure 1) gives a contribution which reads

$$-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}B_\mu(k)B_\nu(-k)\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\frac{-(2p+k)_\mu(2p+k)_\nu M^{AB}(k,p)}{p^2(p+k)^2}\quad , \tag{2.15}$$

where we have introduced the tensor

$$M^{AB}(k,p)=(-d\sin\frac{k\tilde p}{2}+f\cos\frac{k\tilde p}{2})^{ALM}(d\sin\frac{k\tilde p}{2}+f\cos\frac{k\tilde p}{2})^{BML}\quad . \tag{2.16}$$

The second diagram, shown in Figure 2, gives

$$\int\frac{d^4k}{(2\pi)^4}B_\mu^A(k)B_\nu^B(-k)\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\frac{-\delta_{\mu\nu}\ M^{AB}(k,p)}{p^2}\quad . \tag{2.17}$$

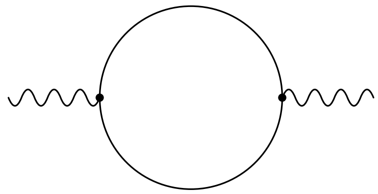

Figure 1.

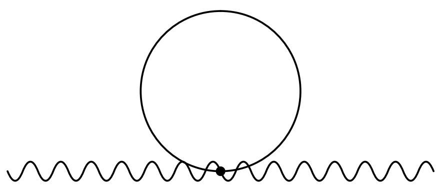

Figure 2.

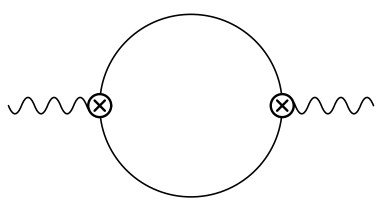

Figure 3.

In (2.15), (2.17) the trace is over spin indices, and its effect leads to a multiplicative factor of

$$\mathrm{Tr}\mathbb{1}_j \equiv d(j) \quad , \tag{2.18}$$

where $d(j)$ is the number of spin component of the field ϕ,

$$d(j) \equiv \quad 1 \quad \text{for scalars,} \qquad 2 \quad \text{for Weyl fermions,} \qquad 4 \quad \text{for vectors.} \tag{2.19}$$

It is worth remarking that in supersymmetric theories the cancellation between bosonic and fermionic degrees of freedom enters the game via the identity

$$\sum_j \alpha_j d(j) = 0 \quad , \tag{2.20}$$

which holds for any representation of the gauge group. This in turn implies that the first and the second diagram separately vanish in any supersymmetric theory, even in presence of (supersymmetry preserving) spontaneous symmetry breaking.

We now move on to the third amplitude, which is depicted in Figure 3 and gives

$$-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} B^A_\mu(k) B^B_\nu(-k) \int \frac{d^Dp}{(2\pi)^D} \mathrm{Tr} \frac{-4J^{\mu\rho}J^{\nu\lambda}k_\lambda k_\rho M^{AB}(k,p)}{p^2(p+k)^2} \quad , \tag{2.21}$$

where in the spin j representation

$$\mathrm{Tr}(J^{\mu\rho}J^{\nu\lambda})_j = C(j)(\delta^{\mu\nu}\delta^{\rho\lambda} - \delta^{\mu\lambda}\delta^{\nu\rho}) \quad , \tag{2.22}$$

$$C(j) \equiv \quad 0 \quad \text{for scalars,} \qquad \tfrac{1}{2} \quad \text{for Weyl fermions,} \qquad 2 \quad \text{for vectors.} \tag{2.23}$$

To proceed further on, we rewrite (2.16) using the relations [17]

$$\begin{aligned} f^{ALM} f^{BML} &= -N c_A \delta_{AB} \quad , \\ d^{ALM} d^{BML} &= N d_A \delta_{AB} \quad , \\ f^{ALM} d^{BML} &= 0 \quad , \end{aligned} \tag{2.24}$$

where $c_A = 1 - \delta_{0A}$ and $d_A = 2 - c_A$. This way (2.16) collapses to

$$M^{AB}(k,p) = -N \ \delta^{AB}(1 - \delta_{0A}\cos k\tilde{p}) \quad . \tag{2.25}$$

The two terms on the right hand side of Eq. (2.25) respectively select the planar and the nonplanar contribution, the latter explicitly depending on the noncommutativity parameter. Using (2.25) we can recast the $U(N)$ polarization tensor as

$$[\Pi^{AB}_{\mu\nu}]^{planar}[U(N)] = N\ \delta^{AB}\Pi^{planar}_{\mu\nu}[U(1)] \quad ,$$

$$[\Pi^{AB}_{\mu\nu}]^{np}[U(N)] = N\ \delta^{A0}\delta^{B0}\Pi^{np}_{\mu\nu}[U(1)] \quad . \tag{2.26}$$

Here $[\Pi^{AB}_{\mu\nu}]^{planar}[U(1)]$ and $\Pi^{np}_{\mu\nu}[U(1)]$ are respectively the planar and nonplanar contributions to the polarization tensor for gauge group $U(1)$, and have been calculated in [9]. In particular $\Pi^{np}_{\mu\nu}[U(1)]$ contains the IR/UV mixing terms characteristic to noncommutative $U(1)$ theories as shown in [9]. Equations (2.26) remarkably show the decoupling of the $U(1)$ component associated with the generator $t^0 \propto \mathbb{1}$, as well as the absence of nonplanar contributions for the $SU(N)$ fields in the effective action,[4]

$$\frac{1}{g^2_{\text{eff}}(k)}_{[N^2]\times[N^2]} = \frac{1}{g'^2_{\text{eff}}(k)} \oplus \frac{1}{g^2_{\text{eff}}(k)}_{[N^2-1]\times[N^2-1]} \tag{2.27}$$

We now move on to consider the spontaneously broken case.

2.3 Spontaneously broken theories

In this subsection we follow the analysis of [19] and focus only on $\mathcal{N} = 2$ noncommutative Super Yang-Mills theories in the Coulomb phase. They can be conveniently described as the dimensional reduction of $\mathcal{N} = 1$ Super Yang-Mills in 6 space-time dimensions down to 4 dimensions [20]. More precisely, we extend the $U(N)$ gauge field A_μ to a 6-dimensional vector field incorporating the two real scalars of pure $\mathcal{N} = 2$ Super Yang-Mills:

$$A^{\text{6D}}_\mu = \big(A_1\,,\,A_2\,,\,A_3\,,\,A_4\,,\,A_5 \equiv \varphi_1\,,\,A_6 \equiv \varphi_2\big) \quad . \tag{2.28}$$

Assuming that all relevant field configurations are independent of the final two compactified spatial directions, then the 4-dimensional $\mathcal{N} = 2$ $U(N)$ Lagrangian is known to be simply that of 6-dimensional $\mathcal{N} = 1$ Super Yang-Mills theory. At the one-loop level we focus on terms quadratic in the fluctuating fields, and replace (2.1) by

$$\begin{aligned} A^{\text{6D}}_\mu &= B_\mu + N_\mu \quad , \quad \mu = 1, \dots, 4 \quad , \\ A^{\text{6D}}_5 &= \mathrm{v}^{(1)} + B_5 + N_5 \quad , \\ A^{\text{6D}}_6 &= \mathrm{v}^{(2)} + B_6 + N_6 \quad . \end{aligned} \tag{2.29}$$

Here $\mathrm{v}^{(1)} + i\mathrm{v}^{(2)} \equiv \mathrm{v}$ is the (constant) vacuum expectation value of the complex scalar field φ of the 4-dimensional $\mathcal{N} = 2$ supersymmetry which minimizes the potential $\mathrm{Tr}([\varphi, \varphi^\dagger]_\star)^2$. As

[4]A similar decoupling between $U(1)$ and $SU(N)$ components was observed in [18] for the one-loop gluon propagator in noncommutative QCD with N colours.

usual, v can be expanded in terms of the Cartan generators H_u, $u = 1, \ldots, N$ of the gauge group G as

$$\mathrm{v} = \sum_{u=1}^{N} \mathrm{v}_u H_u \quad . \tag{2.30}$$

The Higgs mechanism breaks the gauge symmetry to that of the Cartan subalgebra H, *i.e.* $U(1)^N$. However, notice that a noncommutative $U(1)$ theory is not spontaneously broken when its scalar acquires a nonvanishing vacuum expectation value φ_0 [8], as it follows directly from observing that $\Omega\varphi_0\Omega^{-1} = \varphi_0$, where $\Omega \in U(1)_\star$. To see how this circumstance affects the full $U(N)$ theory, let us now move on to the calculation of the effective action. In the one-loop approximation, the only difference with the unbroken case comes from the new terms which appear in the generalized 'kinetic' operator (2.10) as a consequence of expanding the 6-dimensional gauge field as in (2.29) with $\mathrm{v} \neq 0$. Without loss of generality we can set $\mathrm{v}^{(2)} = 0$; it is then immediately realized that, under the hypothesis of independence of the compactified dimensions, the relevant kinetic operators are replaced by

$$\Delta_{j,\mathbf{G}} \star \phi \longrightarrow \Delta_{j,\mathbf{G}} \star \phi - [\mathrm{v}, [\mathrm{v}, \phi]] - 2[\mathrm{v}, [B_5, \phi]_\star]_\star \quad . \tag{2.31}$$

The last term in the right hand side of (2.31) corresponds to a new interaction vertex. However, it is very easy to convince oneself that the corresponding new contributions to the one-loop expansion of the logarithm in (2.9) separately vanish as a consequence of supersymmetry, (2.20), and of the property Tr $[J^{\mu\nu}] = 0$ of the Lorentz generators.

The first term in the right hand side of (2.31) corresponds to a mass term in the tree level action,

$$\int dx \; \mathrm{Tr}(\bar{\phi} \; [\bar{\mathrm{v}}, \; [\mathrm{v}, \phi] \;]) \equiv \int dx \; \bar{\phi}^A \mathcal{M}^2_{AB} \phi^B \quad , \tag{2.32}$$

where the trace is in the group space. Using for the generators of the $U(N)$ algebra a basis $\{H^u \, , E^{uv}_\pm (u > v)\}$, with $H^u_{AB} = \delta^u_A \delta^u_B$, and decomposing accordingly the fields as $\sum_{u=1}^N \phi_u H^u + \phi^\pm_{uv} E^{uv}_\pm$, it is immediately seen that the Higgs mechanism gives the $\phi^\pm_{uv}$ components masses proportional to the differences $|\mathrm{v}_u - \mathrm{v}_v|$. No dependence on the 'center of mass' coordinate $\sum_{u=1}^N \mathrm{v}_u$ appears, which thus does not influence physics. Therefore, in the low-energy effective action N massless supermultiplets are expected, but only $N-1$ moduli.

To efficiently perform perturbative expansions, it is convenient to define an operator $\mathcal{G}$ as the inverse in momentum space of the new tree level kinetic term, *i.e.* $\mathcal{G}^{AB} \equiv [(-\partial^2 + \mathcal{M}^2)^{-1}]^{AB}$; for example, when the gauge group is $U(2)$ we can without loss of generality set $\mathrm{v}^a \propto \delta^{a3}$, and the resulting $\mathcal{G}$ is the diagonal colour-space matrix

$$\mathcal{G}^{AB} \;=\; \mathrm{diag}(\; \mathcal{G}^{00} \, , \mathcal{G}^{ab} \;) \;=\; \mathrm{diag}\Big(\, \frac{1}{p^2} \, , \, \frac{1}{p^2 - M_W^2} \, , \, \frac{1}{p^2 - M_W^2} \, , \, \frac{1}{p^2} \, \Big) \quad . \tag{2.33}$$

The first entry corresponds to the $U(1)$ subgroup associated to the generator t^0, the last to the $a = b = 3$ component and M_W is the mass of the $W^\pm$ bosons, $M_W = \sqrt{2}|\mathrm{v}|$. The one-loop perturbation theory goes on as in the last section, with the only modification of using as propagators the appropriate $U(N)$ generalization of (2.33).

Next we ask whether in the spontaneously broken theory there are new IR/UV mixing effects compared to the unbroken case studied in the previous subsection. To answer this question, we need to look only at the very high loop momentum contribution to the Feynman amplitudes, which is responsible for the interplay between ultraviolet and infrared divergences [5]. In this approximation all the masses and external momenta can be ignored which means that in the IR we get precisely the same decoupling pattern of the $U(1)$ and no new IR/UV mixing effects. Thus, there is no IR/UV mixing in the $SU(N)$ theory, which means that the noncommutative $SU(N)$ behaves in the same way in the IR as its commutative counterpart.

3. Multi-instanton contributions to the prepotential

In this Section we will explain why all the multi-instanton contributions to the prepotential in the noncommutative theory precisely agree with those computed in the ordinary commutative theory.

We consider the $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory directly in Euclidean[5] noncommutative space. It will be convenient to parametrize the six independent components of $\theta^{\mu\nu}$ in terms of the self-dual and the anti-self-dual combinations

$$\zeta^c_{(+)} \equiv \bar\eta^c_{\mu\nu}\theta^{\mu\nu} \ , \qquad \zeta^c_{(-)} \equiv \eta^c_{\mu\nu}\theta^{\mu\nu} \ , \qquad c = 1, 2, 3 \ , \tag{3.1}$$

where $\bar\eta^c_{\mu\nu}$ and $\eta^c_{\mu\nu}$ are the standard self-dual and anti-self-dual 't Hooft symbols. Note that in Euclidean space all the six components of $\{\zeta^c_{(+)}, \zeta^c_{(-)}\}$ are real and independent.

We now turn to the effective Lagrangian (1.7). The general superfield expression on the right hand side of (1.7) can be expanded in component-fields and will contain the characteristic 4-fermion and 4-anti-fermion interactions:

$$\begin{aligned}\mathcal{L}_{\rm eff} \ \ni \ & \frac{1}{32\pi i}\left\{\partial^4_{\mathrm{v}}\mathcal{F}(\mathrm{v}, \zeta_{(+)}, \zeta_{(-)})\,\lambda\star\lambda\star\lambda\star\lambda \ - \ \partial^4_{\bar{\mathrm{v}}}\mathcal{F}(\mathrm{v}, \zeta_{(+)}, \zeta_{(-)})^\dagger\,\bar\lambda\star\bar\lambda\star\bar\lambda\star\bar\lambda\right\} \\ = \ & \frac{1}{32\pi i}\left\{\partial^4_{\mathrm{v}}\mathcal{F}(\mathrm{v}, \zeta_{(+)}, \zeta_{(-)})\,\lambda\star\lambda\star\lambda\star\lambda \ - \ \partial^4_{\bar{\mathrm{v}}}\mathcal{F}^*(\bar{\mathrm{v}}, \zeta_{(+)}, \zeta_{(-)})\,\bar\lambda\star\bar\lambda\star\bar\lambda\star\bar\lambda\right\} \ ,\end{aligned} \tag{3.2}$$

where $\bar\lambda(x)$ and $\lambda(x)$ are the (anti-)gauginos of the $\mathcal{N} = 2$ theory. The dagger, $\mathcal{F}^\dagger$, on the first line of (3.2) denotes the Hermitean conjugation which also conjugates the argument of the

[5]For $\mathcal{N} > 1$ there are no problems with Euclidean formulations of supersymmetry.

function, and the asterisck, $\mathcal{F}^*$, on the second line on (3.2) denotes complex conjugation of the function without complex-conjugating the argument.[6]

It is well-known [13, 21] that instantons contribute to the 4-fermion vertex whereas anti-instantons contribute to the 4-anti-fermion vertex in the effective action:

$$\mathcal{L}_{\rm eff} \ \ni \ G_{\rm inst}\,\lambda\star\lambda\star\lambda\star\lambda \ + \ G_{\text{anti-inst}}\,\bar\lambda\star\bar\lambda\star\bar\lambda\star\bar\lambda \ , \tag{3.3}$$

where $G_{\rm inst}$ and $G_{\text{anti-inst}}$, can be determined from the Green functions

$$\langle\bar\lambda\bar\lambda\bar\lambda\bar\lambda\rangle_{\rm inst} \ , \qquad \langle\lambda\lambda\lambda\lambda\rangle_{\text{anti-inst}} \ ,$$

computed in the instanton and the anti-instanton backgrounds. The prepotential $\mathcal{F}$ can be now recovered in two independent ways. The first is by relating $\partial^4_{\rm v}\mathcal{F}({\rm v},\zeta_{(+)},\zeta_{(-)})$ to $G_{\rm inst}$, the second, by relating $\partial^4_{\bar{\rm v}}\mathcal{F}^*(\bar{\rm v},\zeta_{(+)},\zeta_{(-)})$ to $G_{\text{anti-inst}}$.

In general $G_{\rm inst}$ and $G_{\text{anti-inst}}$ depend on the VEVs and on the noncommutativity parameters. As will be explained in the next Section, the latter dependence is very restrictive: $G_{\rm inst}$ depends on $\zeta_{(+)}$ and not on $\zeta_{(-)}$, and $G_{\text{anti-inst}}$ depends on $\zeta_{(-)}$ and not on $\zeta_{(+)}$. This is the consequence of the fact that the (anti)-self-dual solutions in noncommutative Yang-Mills do not depend on the noncommutativity parameter of the opposite duality [3, 22] as it is immediately realized by looking at the expressions for the ADHM constraints. From this we conclude that $\mathcal{F}$ cannot depend on $\zeta_{(-)}$, and $\mathcal{F}^*$ cannot depend on $\zeta_{(+)}$. This means that $\mathcal{F}$ does not depend neither on $\zeta_{(-)}$, nor on $\zeta_{(+)}$.

In fact it is easy to argue that the prepotential $\mathcal{F}$ in the noncommutative theory can be calculated directly at $\zeta_{(\pm)}=0$ leading to an expression for $\mathcal{F}$ which is identical to the ordinary commutative case. Let us set $\zeta_{(+)}=0$ and keep $\zeta_{(-)}\neq 0$. The instanton contribution is then identical to the commutative theory, as the instanton itself and the instanton measure coincide with their commutative counterparts. The instanton prediction for $\mathcal{F}$ is the same as in the commutative theory and must match with the anti-instanton prediction for $\mathcal{F}$. But the anti-instanton is truly noncommutative, as $\zeta_{(-)}\neq 0$. From this matching it follows that the general noncommutative (anti)-instanton contribution to $\mathcal{F}$ can be computed at $\zeta_{(\pm)}=0$ leading to the ordinary commutative prepotential.

In the next Section we check this general argument against an explicit one-instanton calculation.

[6]For example, if $f(x)=1+ix$, then $f(x)^\dagger=1-i\bar{x}$, and $f^*(x)=1-ix$, such that $f(x)^\dagger=f^*(\bar{x})$.

4. Instanton calculations

In this Section we introduce the necessary tools to perform explicit multi-instanton calculations in noncommutative $\mathcal{N} = 2$ theories. We then evaluate the one-instanton contribution to $\mathcal{F}$ and confirm the general argument presented in the previous Section.

4.1 Multi-instanton supermultiplet

The multi-instanton configuration in the noncommutative $\mathcal{N} = 2$ supersymmetric $U(N)$ Yang-Mills theory on the Coulomb branch is a (constrained) solution of the equations of motion of the theory in noncommutative Euclidean space-time.

Let us first consider a pure noncommutative (nonsupersymmetric) $U(N)$ gauge theory. The k-(anti)-instanton gauge field is the general solution of the (anti)-self-duality equations with instanton charge $\pm k$. This (anti)-self-dual gauge configuration follows from the ADHM analysis [23–25] and can be conveniently parametrized by the $[N+2k]\times[2k]$ matrix of instanton collective coordinates $a_{[N+2k]\times[2k]}$. This matrix can be written as [26]

$$a_{[N+2k]\times[2k]} \;=\; \begin{pmatrix} w_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix} \;=\; \begin{pmatrix} w_{u\,i\dot\alpha} \\ (a'_{\beta\dot\alpha})_{li} \end{pmatrix} \quad . \tag{4.1}$$

In this Section we will closely follow notation and conventions adopted in the Instanton Hunter's Guide [26] to which the reader is referred for more detail on the instanton calculus. In particular we use the following index assignments:

$$\begin{aligned} \text{Instanton number indices } [k]: &\quad 1 \le i, j, l \cdots \le k \\ \text{U(N) Color indices } [N]: &\quad 1 \le u, v \cdots \le N \\ \text{ADHM indices } [N+2k]: &\quad 1 \le \lambda, \mu \cdots \le N+2k \\ \text{Quaternionic (Weyl) indices } [2]: &\quad \alpha, \beta, \dot\alpha, \dot\beta \cdots = 1, 2 \end{aligned}$$

Importantly, not all the components of the ADHM matrix a are independent. Much of this redundancy can be eliminated by noting that the ADHM construction of the gauge field is unaffected by x-independent $U(k)$ transformations of the form

$$w_{ui\dot\alpha} \;\to\; w_{uj\dot\alpha}\,R_{ji}\;, \quad (a'_{\alpha\dot\alpha})_{ij} \;\to\; R^\dagger_{il}\,(a'_{\alpha\dot\alpha})_{lp}\,R_{pj}\;, \quad R_{ij} \in U(k)\;. \tag{4.2}$$

Finally, and most importantly, the self-duality equations require that

$$\mathrm{tr}_2\, \tau^c \bar a a \;=\; 0 \tag{4.3a}$$

$$(a'_n)^\dagger \;=\; a'_n\;. \tag{4.3b}$$

In Eq. (4.3a) we have contracted $\bar{a}^{\dot\beta}a_{\dot\alpha}$ with the three Pauli matrices $(\tau^c)^{\dot\alpha}{}_{\dot\beta}$, while in Eq. (4.3b) we have decomposed $(a'_{\alpha\dot\alpha})_{li}$ and $(\bar{a}'^{\dot\alpha\alpha})_{il}$ in the usual quaternionic basis of spin matrices:

$$(a'_{\alpha\dot\alpha})_{li} \;=\; (a'_\mu)_{li}\,\sigma^\mu_{\alpha\dot\alpha}\;, \qquad (\bar{a}'^{\dot\alpha\alpha})_{il} \;=\; (a'_\mu)_{il}\,\bar\sigma^{\mu\,\dot\alpha\alpha}\;. \tag{4.4}$$

Equation (4.3a) is the famous non-linear matrix equation which is frequently referred to as the ADHM constraint. We can count the independent bosonic collective coordinates of the ADHM multi-instanton solution. A general complex matrix $a_{[N+2k]\times[2k]}$ has $4k(N+2k)$ real degrees of freedom, $n_{\rm b}$. The two ADHM conditions (4.3a) and (4.3b) impose $3k^2$ and $4k^2$ real constraints, respectively, while modding out by the residual $U(k)$ symmetry removes another k^2 degrees of freedom. In total we therefore have

$$n_{\rm b} \;\equiv\; 4k(N+2k) - 3k^2 - 4k^2 - k^2 \;=\; 4kN \tag{4.5}$$

real degrees of freedom, precisely as required.

We now can return to the gauge theory on noncommutative space. The noncommutative multi-instanton configuration can be obtained by a straightforward modification of the ordinary ADHM construction. Nekrasov and Schwarz [22] showed that in noncommutative space the (anti)-instanton ADHM constraint (4.3a) is shifted by the (anti)-self-dual component of $\theta^{\mu\nu}$

$$\text{instanton}\;:\qquad {\rm tr}_2\,(\tau^c\bar{a}a)_{ij} - \zeta^c_{(+)}\delta_{ij} \;=\; 0\;, \qquad \zeta^c_{(+)} \equiv \bar\eta^c_{\mu\nu}\theta^{\mu\nu} \tag{4.6a}$$

$$\text{anti-instanton}\;:\qquad {\rm tr}_2\,(\tau^c\bar{a}a)_{ij} - \zeta^c_{(-)}\delta_{ij} \;=\; 0\;, \qquad \zeta^c_{(-)} \equiv \eta^c_{\mu\nu}\theta^{\mu\nu} \tag{4.6b}$$

In $\mathcal{N}=2$ supersymmetric gauge theories the gauge field is accompanied by two gauginos, λ^A, $A=1,2$ and a complex Higgs field, all in the adjoint representation of $U(N)$. The corresponding instanton component fields were determined in [26]. The instanton components of gauginos are traditionally referred to as the adjoint fermion zero modes. They have an associated set of Grassmann collective coordinates which can be arranged into the $[N+2k]\times[k]$ matrices $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$ as in [26]

$$\mathcal{M}^A_{[N+2k]\times[k]} \;=\; \begin{pmatrix} \mu^A_{ui} \\ (\mathcal{M}'^A_\beta)_{li} \end{pmatrix}\;, \qquad \bar{\mathcal{M}}^A_{[k]\times[N+2k]} \;=\; \begin{pmatrix} \bar\mu^A_{iu}\;, & (\bar{\mathcal{M}}'^{\beta A})_{il} \end{pmatrix}\;. \tag{4.7}$$

Dirac equations for the fermion zero modes in the ADHM background require the matrices $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$ to satisfy the so-called fermionic ADHM constraints:

$$\bar{\mathcal{M}}^A_i\,a_{j\dot\alpha} \;=\; -\bar{a}_{i\dot\alpha}\,\mathcal{M}^A_j\;, \tag{4.8a}$$

$$\bar{\mathcal{M}}'^A_\alpha \;=\; \mathcal{M}'^A_\alpha\;. \tag{4.8b}$$

Equation (4.8b) allows us to eliminate $\bar{\mathcal{M}}'^A$ in favour of $\mathcal{M}'^A$. Counting the number of fermionic degrees of freedom for the first gaugino, λ^1, one finds $2k(N+2k)$ real Grassmann parameters in $\mathcal{M}^1$ and $\bar{\mathcal{M}}^1$, subject to $2k^2$ constraints from each of Eqs. (4.8a), (4.8b) for a net of $2Nk$

gaugino zero modes as required. The same counting applies to the second fermion flavour, λ^2, with the total net effect of

$$n_{\rm f} \;\equiv\; 4Nk \;. \tag{4.9}$$

For future reference we note here, following [26], that the instanton supermultiplet also contains the anti-gaugino components, $\bar\lambda^A$, which, however, do not lead to new Grassmann collective coordinates.

Finally, the adjoint Higgs field configuration is constructed in [26] in terms of the auxiliary $k\times k$ anti-Hermitian matrix $\mathcal{A}_{\rm tot}$ which is defined as the solution to the inhomogeneous linear equation

$$\mathbf{L}\cdot\mathcal{A}_{\rm tot} \;=\; \Lambda_{\rm tot} \;, \tag{4.10}$$

where $\Lambda_{\rm tot}$ is the $k\times k$ anti-Hermitian matrix

$$\Lambda_{{\rm tot}\,ij} \;=\; \bar w^{\dot\alpha}_{iu}\,\langle\mathcal{A}\rangle_{uv}\,w_{vj\dot\alpha} \;+\; \frac{1}{2\sqrt2}\left(\bar{\mathcal{M}}^1\mathcal{M}^2-\bar{\mathcal{M}}^2\mathcal{M}^1\right)_{ij} \;, \tag{4.11}$$

and the $N\times N$ matrix $\langle\mathcal{A}\rangle$ is just i times the VEV matrix,

$$\langle\mathcal{A}\rangle_{uv} \;=\; i\;{\rm diag}({\rm v}_1,\ldots,{\rm v}_N) \;. \tag{4.12}$$

$\mathbf{L}$ is a linear operator that maps the space of $k\times k$ scalar-valued anti-Hermitian matrices onto itself. Explicitly, if Ω is such a matrix, then $\mathbf{L}$ is defined as

$$\mathbf{L}\cdot\Omega \;=\; \tfrac12\{\,\Omega\,,\,W\,\} \;-\; \tfrac12{\rm tr}_2\left([\,\bar a'\,,\,\Omega\,]a'-\bar a'[\,a'\,,\,\Omega\,]\right) \tag{4.13}$$

where W is the Hermitian $k\times k$ matrix $W_{ij}=\bar w^{\dot\alpha}_{iu}w_{uj\dot\alpha}$. Note that the matrix $\mathcal{A}$ is completely determined by the inhomogeneous equation (4.10), as a result, there are no (unconstrained) collective coordinates associated with the Higgs. However, we can still interpret the $\mathcal{A}$ as the (constrained) collective coordinates for the Higgs field, subject to the 'ADHM Higgs constraint' (4.10).

The k-instanton action in the $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory was calculated in [26]. It reads:

$$\begin{aligned} S^{(k)} \;=\;& \frac{8k\pi^2}{g^2} \;+\; 8\pi^2\;\bar w^{\dot\alpha}_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\langle\mathcal{A}\rangle_{uu}w_{ui\dot\alpha} \;-\; 8\pi^2\;\bar w^{\dot\alpha}_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}w_{uj\dot\alpha}(\mathcal{A}_{\rm tot})_{ji} \\ &+\; 2\sqrt2\,\pi^2\big(\bar\mu^1_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\mu^2_{ui} \;-\; \bar\mu^2_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\mu^1_{ui}\big) \;, \end{aligned} \tag{4.14}$$

where $\langle\bar{\mathcal{A}}\rangle_{uv} \;=\; -i\;{\rm diag}(\bar{\rm v}_1,\ldots,\bar{\rm v}_N)$

Similar considerations apply to the anti-instanton supermultiplet, which will have fermion and anti-fermion components interchanged, and $\langle\mathcal{A}\rangle$ exchanged with $\langle\bar{\mathcal{A}}\rangle$. The anti-instanton action is then

$$\begin{aligned} S^{(-k)} &= \frac{8k\pi^2}{g^2} + 8\pi^2\, \bar{w}_{iu}^{\dot{\alpha}} \langle\bar{\mathcal{A}}\rangle_{uu} \langle\mathcal{A}\rangle_{uu} w_{ui\dot{\alpha}} - 8\pi^2\, \bar{w}_{iu}^{\dot{\alpha}} \langle\bar{\mathcal{A}}\rangle_{uu} w_{uj\dot{\alpha}} (\mathcal{A}_{\rm tot})_{ji} \\ &+ 2\sqrt{2}\,\pi^2 \big(\bar{\mu}_{iu}^1 \langle\mathcal{A}\rangle_{uu} \mu_{ui}^2 - \bar{\mu}_{iu}^2 \langle\mathcal{A}\rangle_{uu} \mu_{ui}^1\big) , \end{aligned} \tag{4.15}$$

where the Grassmann collective coordinates μ, $\bar{\mu}$ and $\mathcal{M}'$ correspond to the antifermion zero modes.

4.2 Multi-(anti)-instanton measure

The collective-coordinate $(\pm k)$-instanton integration measure $d\mu^{(\pm k)}$ for a noncommutative $\mathcal{N}=2$ supersymmetric $U(N)$ Yang-Mills on the Coulomb branch is easily obtained from the measure in the ordinary commutative theory, derived in [26].

It can be argued in parallel with Seiberg and Witten [3], that the only effect of noncommutativity on the (anti)-instanton measure and the action is the shift of the gauge-field-ADHM constraint (4.3a) as described by (4.6a) and (4.6b). In particular, in the noncommutative $\mathcal{N}=2$ supersymmetric theory, the measure has the following form cf. [26]:

$$\begin{aligned} \int d\mu^{(\pm k)}\ \exp[-S^{(\pm k)}] &= \frac{M_{PV}^{2Nk}(C_1')^k}{\mathrm{Vol}\,U(k)} \int d^{4k^2}a'\ d^{2kN}\bar{w}\ d^{2kN}w \prod_{A=1,2} d^{2k^2}\mathcal{M}'^A\ d^{kN}\bar{\mu}^A\ d^{kN}\mu^A \\ &\times\ d^{k^2}\mathcal{A}_{\rm tot} \prod_{c=1,2,3} \delta^{(k^2)}\big(\tfrac{1}{2}(\mathrm{tr}_2\ \tau^c\bar{a}a - \zeta_{(\pm)})\big) \prod_{A=1,2} \delta^{(2k^2)}(\bar{\mathcal{M}}^A a + \bar{a}\mathcal{M}^A) \\ &\times \delta^{(k^2)}(\mathbf{L}\cdot\mathcal{A}_{\rm tot} - \Lambda_{\rm tot})\ \exp[-S^{(\pm k)}] . \end{aligned} \tag{4.16}$$

Here the integrals on the right hand side of (4.16) are over all the collective coordinates of the instanton supermultiplet. The ADHM constraints for gauge field (4.6a) and (4.6b), fermions (4.8a), and the Higgs (4.10), are explicitly implemented via the delta-functions.

We stress that the noncommutativity parameters $\zeta_{(\pm)}$ appears in (4.16) only via the expression $\delta^{(k^2)}\big(\frac{1}{2}(\mathrm{tr}_2\ \tau^c\bar{a}a - \zeta_{(\pm)})\big)$. All the other factors in the instanton partition function (4.16) (including the other constraints and expression for the instanton action (4.14)) are unchanged. One way to understand this is to appeal to the k-D-instanton partition function in the presence of N D3-branes in type IIB string theory, which was derived in Section IV.2 of [28]. In the $\alpha' \to 0$ limit this D-instanton partition function reduces to the Yang-Mills-instanton partition function in $\mathcal{N}=4$ supersymmetric Yang-Mills on the world-volume of D3-branes. As explained

in [3], the noncommutativity on the world-volume of D3-branes is introduced by turning on a background $B_{\mu\nu}$ field. In the k-D-instanton matrix theory this corresponds to turning on the Fayet–Iliopolos (FI) couplings in the $U(1)$ subgroup of the $U(k)$ gauge theory, which leads precisely to the modification of the gauge-field-ADHM constraints (4.6a) and (4.6b). Calculations of instanton partition functions in $\mathcal{N}=4$ with and without noncommutativity were performed in [29]. Finally, $\mathcal{N}=4$ supersymmetry can be then softly broken by mass terms to $\mathcal{N}=2$, leading precisely [28] to (4.16).

The factor of C_1' on the right hand side of (4.16) is a numerical constant associated with the normalization of the 1-instanton measure, and M_{PV}^{2Nk} corresponds to the Pauli-Villars regulator to the power $n_{\rm b}-\frac{1}{2}n_{\rm f}=2Nk$ which arises from the ratio of the UV-regularized bosonic and fermionic fluctuation determinants. Combined with $\exp[-8k\pi^2/g^2(M_{PV})]$ from the instanton action it gives rise to the renormalization group invariant scale Λ_{PV} of the $\mathcal{N}=2$ noncommutative $U(N)$ theory,

$$M_{PV}^{2Nk}\ \exp\left[-\frac{8k\pi^2}{g^2(M_{PV})}\right]\ =\ \Lambda_{PV}^{2Nk}\ \equiv\ \Lambda_{PV}^{b_0 k}\ . \tag{4.17}$$

4.3 Explicit expression for the $\mathcal{N}=2$ prepotential

The general expression for the k-instanton contribution to the prepotential (1.9) was derived in [21, 26, 27]

$$\mathcal{F}^{(k)}(\mathrm{v})\ =\ 8\pi i\int d\tilde{\mu}^{(k)}\ \exp[-S^{(k)}] \tag{4.18}$$

Here $d\tilde{\mu}^{(k)}$ is the "reduced measure" which is obtained from the $\mathcal{N}=2$ measure, $d\mu^{(k)}$, as follows:

$$\int d\mu^{(k)}\ =\ \int d^4x_0\, d^2\xi_1\, d^2\xi_2\int d\tilde{\mu}^{(k)}\ , \tag{4.19}$$

where (x_0,ξ_1,ξ_2) gives the global position of the multi-instanton in $\mathcal{N}=2$ superspace. Explicitly, the instanton center x_0^μ and the supersymmetric fermion zero modes ξ_1, ξ_2 are the linear combinations proportional to the "trace" components of the $k\times k$ matrices a', $\mathcal{M}'^1$ and $\mathcal{M}'^2$, respectively:

$$x_0\ =\ \frac{1}{k}\mathrm{Tr}_k\, a'\ ,\quad \xi_1\ =\ \frac{1}{4k}\mathrm{Tr}_k\mathcal{M}'^1\ ,\quad \xi_2\ =\ \frac{1}{4k}\mathrm{Tr}_k\mathcal{M}'^2\ . \tag{4.20}$$

Note that these $\mathcal{N}=2$ superspace modes do not enter into the δ-function constraints and so do indeed factor out in this simple way. Furthermore, the four exact supersymmetric fermion zero modes $\xi_{1\alpha}$ and $\xi_{2\alpha}$ are the only fermionic modes that are not lifted by (*i.e.* do not appear in) the action (4.14).

Given these expressions for the prepotential, one also knows the all-instanton-orders expansion of the coordinate on the vacuum moduli space of the theory $u_2 = \langle \mathrm{Tr}\, A^2 \rangle$, since on general grounds

$$u_2(\mathrm{v})\Big|_{k\text{-inst}} = 2i\pi k \cdot \mathcal{F}^{(k)}(\mathrm{v}) \ . \tag{4.21}$$

This relation was originally derived by Matone [30], further studied at the 2-instanton level by [31], and the all-instanton-orders proof of it was presented in [27]. The above collective coordinate integral expression for $\mathcal{F}^{(k)}$ constitutes a closed series solution, in quadratures, of the low-energy dynamics of the Coulomb branches of the $\mathcal{N} = 2$ models. It is noteworthy that this solution is obtained purely from the instanton physics.

4.4 One-instanton contribution to the prepotential

In this section we will explicitly evaluate the 1-instanton contribution to the prepotential, $\mathcal{F}^{(1)}$ in the noncommutative theory with non-vanishing ζ_+. We will see that this expression will be exactly the same as in the commutative theory, in agreement with the general argument of the previous section.

Our starting point is the integral expression (4.18) in terms of the reduced noncommutative instanton measure. Our analysis will follow closely the commutative instanton calculation presented in Section 8 of [26]. To evaluate the integral on the right hand side of (4.18), it is helpful to exponentiate the various δ-functions by means of Lagrange multipliers, and to interchange the resulting order of integration. In other words, one integrates out the ADHM supermultiplet $\{a, \mathcal{M}^1, \mathcal{M}^2, \mathcal{A}\}$ first, and only then performs the integration over the Lagrange multipliers.

In the 1-instanton case the spin-1 and spin-1/2 ADHM constraints involve only the top-row elements of matrices a and $\mathcal{M}$. They can be exponentiated in a simple way

$$\prod_{c=1,2,3} \delta\big(\tfrac{1}{2}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\, \bar w_u^{\dot\beta} w_{u\dot\alpha} - \tfrac{1}{2}\zeta^c_{(+)}\big) = \frac{1}{\pi^3} \int d^3\mathbf{p}\, \exp(ip^c(\tau^c \bar w_u w_u - \zeta^c_{(+)}) \ , \tag{4.22}$$

and

$$\prod_{\dot\alpha=1,2} \delta\big(\bar\mu_u^1 w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u^1\big) = 2\int d^2\xi\, \exp\left(\xi^{\dot\alpha}(\bar\mu_u^1 w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u^1)\right) \tag{4.23a}$$

$$\prod_{\dot\alpha=1,2} \delta\big(\bar\mu_u^2 w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u^2\big) = 2\int d^2\eta\, \exp\left(\eta^{\dot\alpha}(\bar\mu_u^2 w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u^2)\right) \tag{4.23b}$$

In this way we introduce the triplet of bosonic Lagrange multipliers p^c, as well as the Grassmann spinor Lagrange multipliers $\xi^{\dot\alpha}$ and $\eta^{\dot\alpha}$. The exponentiation of the spin-0 constraint is best accomplished involving a term in the action (4.14) $8\pi^2\,\bar w_u\langle\bar{\mathcal A}\rangle_{uu}w_u\mathcal A_{\rm tot}\equiv 8\pi^2\bar\Lambda\mathcal A_{\rm tot}$ as follows:

$$\begin{aligned}\int d\mathcal A_{\rm tot}\,\delta(\mathbf L\cdot\mathcal A_{\rm tot}-\Lambda_{\rm tot})\,\exp(8\pi^2\bar\Lambda\mathcal A_{\rm tot}) \;&=\; \frac{1}{\det\mathbf L}\,\exp(8\pi^2\bar\Lambda\cdot\mathbf L^{-1}\cdot\Lambda)\\ =\;8\pi\int d({\rm Re}\,z)d({\rm Im}\,z)\,\exp\big(-8\pi^2(\bar z\,\mathbf L\,z-\bar\Lambda z-\bar z\Lambda_{\rm tot})\big)&\end{aligned}\tag{4.24}$$

The second equality follows from the general Gaussian identity

$$\int\prod_i d({\rm Re}\,z_i)d({\rm Im}\,z_i)\,\exp\big(-\bar z_iK_{ij}z_j+\bar y_iz_i+\bar z_iy_i\big)\;=\;\frac{1}{\det(K/\pi)}\,\exp(\bar y_iK^{-1}_{ij}y_j)\tag{4.25}$$

which can be used to exponentiate the spin-0 constraint in an elegant way for arbitrary instanton number k. The advantage of the rewrite (4.24) is that $\mathbf L$ is easier to manipulate in the exponent than $\mathbf L^{-1}$ (which appears implicitly in the definition of $\mathcal A_{\rm tot}$). In the present case, with $k=1$, the operator $\mathbf L$ collapses to a 1×1 c-number matrix:

$$\mathbf L\;=\;\det\mathbf L\;=\;\bar w_u^{\dot\alpha}w_{u\dot\alpha}\equiv 2\rho^2\ ,\tag{4.26}$$

where ρ is the instanton size. Likewise $\bar\Lambda$ and $\Lambda_{\rm tot}$ are given by

$$\bar\Lambda=-i\bar{\rm v}_u\bar w_u^{\dot\alpha}w_{u\dot\alpha}\ ,\qquad \Lambda_{\rm tot}=i{\rm v}_u\bar w_u^{\dot\alpha}w_{u\dot\alpha}-\frac{1}{2\sqrt2}(\bar\mu_u^2\mu_u^1-\bar\mu_u^1\mu_u^2)\ .\tag{4.27}$$

Now we can perform the Grassmann integrations over $\{\mu_u^1,\mu_u^2,\bar\mu_u^1,\bar\mu_u^2\}$. Consider the combined exponent formed from Eqs. (4.33)-(4.24) and the remaining terms in the 1-instanton action,

$$\exp[-S^{(1)}]\ni\exp\big[-8\pi^2|{\rm v}_u|^2\bar w_u^{\dot\alpha}w_{u\dot\alpha}+2\sqrt2\,\pi^2 i(\bar\mu_u^1\bar{\rm v}_u\mu_u^2-\bar\mu_u^2\bar{\rm v}_u\mu_u^1)\big]\ .\tag{4.28}$$

To eliminate the linear terms in Grassmann variables in the combined exponent, we first perform the linear shifts

$$\begin{aligned}\mu_u^1\to\mu_u^1+\frac{i\eta^{\dot\alpha}w_{u\dot\alpha}}{2\sqrt2\,\pi^2\bar\alpha_u},&\quad \bar\mu_u^1\to\bar\mu_u^1+\frac{i\eta^{\dot\alpha}\bar w_{u\dot\alpha}}{2\sqrt2\,\pi^2\bar\alpha_u},\\ \mu_u^2\to\mu_u^2-\frac{i\xi^{\dot\alpha}w_{u\dot\alpha}}{2\sqrt2\,\pi^2\bar\alpha_u},&\quad \bar\mu_u^2\to\bar\mu_u^2-\frac{i\xi^{\dot\alpha}\bar w_{u\dot\alpha}}{2\sqrt2\,\pi^2\bar\alpha_u}\end{aligned}\tag{4.29}$$

and then perform straightforward integrations over the remaining quadratic terms. By inspection, the Grassmann integrations simply bring down a factor of

$$\prod_{u=1}^N(2\sqrt2\,\pi^2 i\bar\alpha_u)^2\tag{4.30}$$

In (4.29)-(4.30), we have defined α_u and $\bar{\alpha}_u$ as the naturally appearing linear combinations

$$\alpha_u \;=\; \mathrm{v}_u + iz \ , \qquad \bar{\alpha}_u \;=\; \bar{\mathrm{v}}_u - i\bar{z} \ . \tag{4.31}$$

Next, the $\{w_u, \bar{w}_u\}$ integrations are accomplished, using the identity

$$\int d^2 w_u d^2 \bar{w}_u \, \exp\big(- A^0 \bar{w}_u^{\dot{\alpha}} w_{u\dot{\alpha}} + i \sum_{c=1,2,3} A^c (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \, \bar{w}_u^{\dot{\beta}} w_{u\dot{\alpha}} \big) \;=\; \frac{-4\pi^2}{(A^0)^2 + \sum (A^c)^2} \ . \tag{4.32}$$

In this way, all the original ADHM variables $\{a, \mathcal{M}, \mathcal{N}, \mathcal{A}_{\rm tot}\}$ are eliminated from the integral. One is left with an integral over Lagrange multipliers only

$$\mathcal{F}_1 \;= i \frac{C_1'}{2\pi^2} \int d^3\mathbf{p} \, d^2\xi \, d^2\eta \, d(\mathrm{Re}\, z) d(\mathrm{Im}\, z) \, \mathcal{B} e^{-i\mathbf{p}\cdot\boldsymbol{\zeta}} \ , \tag{4.33}$$

where

$$\mathcal{B} = \prod_{u=1}^{N} \frac{(2\sqrt{2}\,\pi^2 i \bar{\alpha}_u)^2 (-4\pi^2)}{\big(8\pi^2 |\alpha_u|^2\big)^2 + \sum_{c=1,2,3} (p^c + \Xi_u^c)^2} \tag{4.34}$$

and Ξ_u^c is the fermion bilinear

$$\Xi_u^c = \frac{1}{4\sqrt{2}\,\pi^2 \bar{\alpha}_u} \big(\xi_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \, \eta^{\dot{\beta}} - \eta_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \, \xi^{\dot{\beta}} \big) \ . \tag{4.35}$$

The expression (4.33) is analogous to Eq. (8.13) of [26], except here we are taking the pure gauge case, $N_F = 0$, and there is an additional phase $e^{-i\mathbf{p}\cdot\boldsymbol{\zeta}}$ that arises from the noncommutativity parameter, $\boldsymbol{\zeta} \equiv \{\zeta^1_{(+)}, \zeta^2_{(+)}, \zeta^3_{(+)}\}$.

The $\{\xi, \eta\}$ Grassmann integrations in (4.33) must be saturated with two insertions of Ξ:

$$\int d^2\xi d^2\eta \, \Xi_u^b \, \Xi_v^c \;=\; \frac{\delta^{bc}}{16\pi^4 \bar{\alpha}_u \bar{\alpha}_v} \ . \tag{4.36}$$

Extracting these quadratic powers of Ξ from $\mathcal{B}$ can be done quite elegantly, thanks to the algebraic identity

$$\begin{aligned} \int d^2\xi d^2\eta \, \mathcal{B} \;&=\; \sum_{b,c=1}^{3} \sum_{u,v=1}^{N} \frac{\delta^{bc}}{16\pi^4 \bar{\alpha}_u \bar{\alpha}_v} \cdot \frac{1}{2} \frac{\partial^2}{\partial \Xi_u^b \, \partial \Xi_v^c} \mathcal{B} \Big|_{\Xi=0} \\ &=\; \frac{1}{32\pi^4 |\mathbf{p}|^2} \Big(\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \Big)^2 \mathcal{B} \Big|_{\Xi=0} \ . \end{aligned} \tag{4.37}$$

Pulling the VEV derivatives outside the integral, one therefore finds:

$$\mathcal{F}_1 \;=\; \frac{iC_1'}{2\pi^2} \cdot \frac{1}{32\pi^4} \Big(\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \Big)^2 \int d(\mathrm{Re}\, z) d(\mathrm{Im}\, z) \, \Gamma \ . \tag{4.38}$$

Here

$$\Gamma \;=\; \int d^3\mathbf{p}\, \frac{e^{-i\mathbf{p}\cdot\boldsymbol{\zeta}}}{|\mathbf{p}|^2} \prod_{u=1}^{N} \frac{(2\sqrt{2}\,\pi^2 i\bar{\alpha}_u)^2(-4\pi^2)}{\left(8\pi^2|\alpha_u|^2\right)^2 + |\mathbf{p}|^2}\;. \tag{4.39}$$

the angular integrals of $\mathbf{p}$ are easily done,

$$\int d(\cos\theta)\, d\phi\, e^{-i\mathbf{p}\cdot\boldsymbol{\zeta}} = \frac{4\pi\sin(\zeta p)}{\zeta p}\;, \tag{4.40}$$

where $p \equiv |\mathbf{p}|$ and $\zeta \equiv |\boldsymbol{\zeta}|$. The integral over p can now be performed as a standard contour integration, extended to run from $-\infty$ to ∞:

$$\Gamma = 16\pi^4\Big(\frac{\pi^2}{2}\Big)^N \frac{1}{8\pi^2\zeta}\sum_{u=1}^{N}\frac{1}{\alpha_u^2}\Big(\prod_{v\neq u}\frac{1}{\alpha_v^2} - e^{-8\pi^2|\alpha_u|^2\zeta}\prod_{v\neq u}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4-|\alpha_u|^4}\Big)\;. \tag{4.41}$$

In this fashion, the original expression (4.33) has collapsed to a 2-dimensional integral over the xy plane (with $x = \mathrm{Re}z$ and $y = \mathrm{Im}z$ henceforth). The remaining integral may be calculated along the lines of [26] and so we follow that approach almost verbatim. Notice that the only dependence on $\bar{\mathrm{v}}_u$ in the integrand is through the variables $\bar{\alpha}_u = \bar{\mathrm{v}}_u - i\bar{z}$. Therefore, it is tempting—but incorrect—to pull the $\bar{\mathrm{v}}_u$ derivatives back inside the integrand, and to make the naive replacement

$$\sum_{u=1}^{N}\frac{\partial}{\partial\bar{\mathrm{v}}_u} \;\to\; i\,\frac{\partial}{\partial\bar{z}}\;, \qquad \Big(\sum_{u=1}^{N}\frac{\partial}{\partial\bar{\mathrm{v}}_u}\Big)^2 \;\to\; -\Big(\frac{\partial}{\partial\bar{z}}\Big)^2\;. \tag{4.42}$$

The error here is due to the fact that the two sides of Eq. (4.42) can differ by δ-function contributions which arise at the locations of poles in the z variable. As a simple example, whereas obviously $\big(\sum \partial/\partial\bar{\mathrm{v}}\big)\, z^{-1} = 0$, one also has, in contrast,[7]

$$\frac{\partial}{\partial\bar{z}}\,\frac{1}{z} \;=\; \pi\,\delta(x)\delta(y)\;, \tag{4.43a}$$

$$\Big(\frac{\partial}{\partial\bar{z}}\Big)^2\frac{1}{z} \;=\; \pi\,\frac{\partial}{\partial\bar{z}}\,\delta(x)\delta(y) \;=\; \frac{\pi}{2}\,\big(\delta'(x)\delta(y) + i\delta(x)\delta'(y)\big)\;. \tag{4.43b}$$

The lesson is that one can legitimately trade $\bar{\mathrm{v}}_u$ differentiation for $\bar{z}$ differentiation as per Eq. (4.42)—but only if one explicitly subtracts off the extraneous δ-function pieces that are generated at the locations of the poles in z. Accordingly, we can split up $\mathcal{F}_1$ into two parts,

$$\mathcal{F}_1 \;=\; \mathcal{F}_\delta \;+\; \mathcal{F}_\partial\;, \tag{4.44}$$

[7] The normalization factor on the right-hand side of Eq. (4.43a) is easily fixed by integrating both sides against $\exp(-\lambda z\bar{z})$.

where $\mathcal{F}_\delta$ is the contribution of these δ-function corrections, while $\mathcal{F}_\partial$ is a boundary term arising from judicious use of Stokes' theorem applied to $\partial^2/\partial\bar{z}^2$. Let us evaluate each of these parts, in turn:

As stated, to calculate $\mathcal{F}_\delta$, one converts $(\sum \partial/\partial\bar{\mathrm{v}}_u)^2$ into $-\partial^2/\partial\bar{z}^2$ as per Eq. (4.42), then subtracts off the spurious δ-function contributions that correspond to the poles in z of the expression Γ given in Eq. (4.41). The relevant poles lie at the N distinct values

$$0 \;=\; \alpha_u \;= \mathrm{v}_u + iz \;=\; (\mathrm{Re}\,\mathrm{v}_u - y) + i(\mathrm{Im}\,\mathrm{v}_u + x)\;. \tag{4.45}$$

There also appear to be poles in Γ when $|\alpha_v|^2 = \pm|\alpha_u|^2$ but these are irrelevant: the poles at $|\alpha_v|^2 = -|\alpha_u|^2$ lie away from the real domain of integration $(x,y) \in \mathbb{R}^2$, whereas the poles at $|\alpha_v|^2 = +|\alpha_u|^2$ have residues that cancel pairwise among the terms in Eq. (4.41) (these pairs correspond to interchanging the indices u and v). In the vicinity of the the singularity (4.45), we have

$$\frac{1}{8\pi^2\zeta\alpha_u^2}\Big(\prod_{v\neq u}\frac{1}{\alpha_v^2} - e^{-8\pi^2|\alpha_u|^2\zeta}\prod_{v\neq u}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4}\Big) \sim \frac{\bar{\alpha}_u}{\alpha_u}\prod_{v\neq u}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4} + \cdots\;, \tag{4.46}$$

which is identical to the behaviour in the case when $\zeta = 0$. In other words this means that $\mathcal{F}_\delta$ is identical to the expression derived in [26] for the $\zeta = 0$ case:

$$\mathcal{F}_\delta \;=\; -\frac{iC_1'\pi^{2N-1}}{2^{N+2}}\sum_{u=1}^{N}\prod_{v\neq u}\frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2}\;. \tag{4.47}$$

Next we consider the boundary term $\mathcal{F}_\partial$ implied by the naive replacement (4.42). It is useful to switch to polar coordinates, $(x,y) \longrightarrow (r,\theta)$, in terms of which

$$\frac{\partial^2}{\partial\bar{z}^2} \;=\; \frac{1}{r}\frac{\partial}{\partial r}\circ\mathcal{D}_r \;+\; \frac{\partial}{\partial\theta}\circ\mathcal{D}_\theta \tag{4.48}$$

where

$$\mathcal{D}_r \;=\; \tfrac{1}{4}e^{2i\theta}\big(2 + r\frac{\partial}{\partial r}\big)\;, \qquad \mathcal{D}_\theta \;=\; \frac{i}{4r^2}\,e^{2i\theta}\big(1 + 2r\frac{\partial}{\partial r} + i\,\frac{\partial}{\partial\theta}\big)\;. \tag{4.49}$$

Since the integrand in Eq. (4.38) is a single-valued function of θ, the $(\partial/\partial\theta)\,\mathcal{D}_\theta$ term can be neglected. Stokes' theorem then equates the 2-dimensional integral (4.38) to the angularly integrated action of $\mathcal{D}_r$ evaluated on the circle of infinitely large radius:

$$\begin{aligned}\mathcal{F}_\partial \;=\; &-\frac{iC_1'}{2\pi^2}\cdot\frac{1}{32\pi^4}\cdot 8\pi^6\lim_{r\to\infty}\tfrac{1}{4}\big(2 + r\frac{\partial}{\partial r}\big)\\ &\times\int_0^\infty d\theta\, e^{2i\theta}\Big[\sum_{u=1}^{N}\frac{1}{8\pi^2\zeta\alpha_u^2}\Big(\prod_{v\neq u}\frac{1}{\alpha_v^2} - e^{-8\pi^2|\alpha_u|^2\zeta}\prod_{v\neq u}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4}\Big)\Big]\;,\end{aligned} \tag{4.50}$$

where $\alpha_u = \mathrm{v}_u + ire^{i\theta}$ and $\bar{\alpha}_u = \bar{\mathrm{v}}_u - ire^{-i\theta}$. But for large r,

$$\frac{1}{8\pi^2\zeta\alpha_u^2}\Big(\prod_{v\neq u}\frac{1}{\alpha_v^2} - e^{-8\pi^2|\alpha_u|^2\zeta}\prod_{v\neq u}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4-|\alpha_u|^4}\Big) \sim r^{-2} \tag{4.51}$$

and therefore $\mathcal{F}_\partial$ vanishes. This is identical to the value of $\mathcal{F}_\partial$ in the case when $\zeta = 0$ [26].

So finally we have proved

$$\mathcal{F}_1 \;\equiv\; \mathcal{F}_\delta \;=\; -\frac{iC_1'\pi^{2N-1}}{2^{N+2}}\sum_{u=1}^{N}\prod_{v\neq u}\frac{1}{(\mathrm{v}_v-\mathrm{v}_u)^2} \tag{4.52}$$

and in particular

$$\mathcal{F}_1(\mathrm{v}_u,\boldsymbol{\zeta}) = \mathcal{F}_1(\mathrm{v}_u,\boldsymbol{\zeta}=0)\ . \tag{4.53}$$

The fact that derivatives of the prepotential are independent of $\boldsymbol{\zeta}$ is a strong constraint on the multi-instanton contributions. In fact it suggests

$$\mathcal{F}_k(\mathrm{v}_u,\boldsymbol{\zeta}) = \mathcal{F}_k(\mathrm{v}_u,0) + \mathcal{S}_k(\zeta)\ , \tag{4.54}$$

where $\mathcal{S}_k$ is independent of the VEVs v_u. This is a very intriguing relation and suggests the following interpretation. It is well known that the effect of the FI term on instantons to modify the instanton moduli space by smoothing out the singularities corresponding to small instantons. For instance for a single instanton in $SU(2)$, the centered moduli space is the singular orbifold $\mathbb{R}^4/\mathbb{Z}_2$, where the radial coordinate is the instanton size and the S^3 is the $SU(2)$ orientation of the instanton. With the FI coupling turned on, the moduli space is smoothed to the Eguchi-Hanson space. Since the prepotential involves an integral over the instanton moduli space, the difference between the prepotential in the commuting and non-commuting theories loosely speaking involves the contribution from the small instanton singularities. The interpretation of (4.54) is then that small instantons are insensitive to the VEV, as is clear from the form of the instanton action, and therefore the contribution from the singularities will be VEV independent. In fact, the result suggests that the only contribution comes from the singularity where all the instantons shrink to zero size at the same point in space. Very similar ideas have been described in the context of the $\mathcal{N} = 4$ theory in [29]. Obviously our discussion here is at best schematic; however, our explicit one instanton calculation provides some supporting evidence where in this case $\mathcal{S}_1 = 0$.

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